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Global existence of classical solutions to the Cauchy problem for a kind of quasilinear hyperbolic systems in divergence form ☆

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ABSTRACT

This paper deals with the global existence of classical solutions to a kind of second order quasilinear hyperbolic systems subject to a null condition, with the linear elastodynamic system as its principal part and the nonlinear terms depending on the product of u^2 and the derivatives of u.

1. Introduction and main result

It is well known that classical solutions to quasilinear hyperbolic equations generically blow up in a finite time even for smooth and small initial data (see [4,5,16]). However, when the quadratic nonlinear terms on the right-hand side of equations satisfy some kind of nonresonance conditions, namely, some kind of so-called null conditions, the problem admits a global smooth solution for small initial data. There have been many results on this topic for nonlinear wave equations (see [2,6,8,11,15,17]). T.C. Sideris and R. Agemi discussed, respectively, the Cauchy problem for nonlinear elastodynamic system with the nonlinearity which satisfies the null conditions and showed independently the global existence of the solution (see [1] and [13]).

Generally speaking, the nonlinear terms only involving derivatives of the unknown functions can be easily handled by using the energy method, however, the nonlinear terms depending on the unknown function, are very difficult to be dealt with. There are some works in this direction. In the case of single wave equation, in order to deal with the nonlinear terms independent of the derivatives of the unknown functions, D. Christodoulou [2] and S. Klainerman [8] adopted an adapted energy inequality which resembles the work of C.S. Morawetz [12]. This approach essentially relies on the Lorentz invariance of the wave operator. In the case of multiple-speed wave equations, J. Metcalfe et al. [11] established a lower regularity decay of solution by means of a pointwise estimate which is similar to the one established by K. Kubota and K. Yokoyama [9], combined with the pointwise estimates of Keel et al. [7] and Huygens' principle. These improved estimates allow them to handle the cubic terms without derivatives.

In this paper, we consider the global existence of classical solutions to a kind of second order quasilinear hyperbolic system subject to a null condition, with the linear elastodynamic system as its principal part and the nonlinear terms in

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divergence form depending on the product of u^2 and the derivatives of u. Comparing with the case considered in [13], the difficulty in this case comes from the terms involving the product of u^2 and the derivatives of u. In order to estimate u, noting that the nonlinear terms are of divergence form, we will use the operators $|D|$, $|D|^{-1}$, the corresponding nonlocal energy $\mathcal{E}(u(t))$ (introduced by Sideris [14]) and the weighted L^2-norm $J_k(u(t))$ (introduced by Y. Du and Y. Zhou [3]). Using the generalized energy estimations, we establish a series of differential inequalities, and then we show the global existence of classical solutions to the problem for small initial data.

We consider the Cauchy problem for the following quasilinear hyperbolic systems

$$\begin{cases} Lu = F\big(u, du, d^2u\big), \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)^T$, $Lu = \partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\nabla \otimes \nabla u$ with wave speeds $0 < c_2 < c_1$ and

$$F\big(u, du, d^2u\big) = N(u,u) + G(u,u,u) + P(u,u,u) + Q(u,u,u) + H(u,u,u), \tag{1.2}$$

where

$$N^i(u,u) = \sum_{\substack{1\leqslant j,k\leqslant 3\\ 1\leqslant l,m,n\leqslant 3}} B^{ijk}_{lmn}\partial_l\big(\partial_m u^j \partial_n u^k\big) \quad (i=1,2,3), \tag{1.3}$$

$$G^i(u,u,u) = \sum_{\substack{1\leqslant j,k,l,m\leqslant 3\\ 0\leqslant \alpha,\beta,\gamma\leqslant 3}} G^{ijkm}_{l\alpha\beta\gamma}\partial_l\big(\partial_\alpha u^j \partial_\beta u^k \partial_\gamma u^m\big) \quad (i=1,2,3), \tag{1.4}$$

$$P^i(u,u,u) = \sum_{\substack{1\leqslant j,k,l,m\leqslant 3\\ 0\leqslant \beta,\gamma\leqslant 3}} P^{ijkm}_{l\beta\gamma}\partial_l\big(u^j \partial_\beta u^k \partial_\gamma u^m\big) \quad (i=1,2,3), \tag{1.5}$$

$$Q^i(u,u,u) = \sum_{\substack{1\leqslant j,k,l,m\leqslant 3\\ 0\leqslant \gamma\leqslant 3}} Q^{ijkm}_{l\gamma}\partial_l\big(u^j u^k \partial_\gamma u^m\big) \quad (i=1,2,3), \tag{1.6}$$

$$H^i(u,u,u) = \sum_{l=1}^{3} \partial_l M^i_l(u,u,u) \quad (i=1,2,3) \tag{1.7}$$

with $M^i_l(u,u,u) = O(|u|^3)$. Here and hereafter, we use the notation $x_0 = t$ and $\partial_0 = \partial_t$ for convenience, and $du = u' = \nabla_{t,x}u$ stands for the space–time gradient. Moreover, the constant coefficients in (1.3)–(1.6) satisfy the following symmetry conditions

$$B^{ijk}_{lmn} = B^{ikj}_{lnm} = B^{jik}_{mln}, \tag{1.8}$$

$$G^{ijkm}_{l\alpha\beta q} = G^{mjki}_{q\alpha\beta l}, \qquad G^{ijkm}_{l\alpha\beta 0} = G^{mjki}_{l\alpha\beta 0}, \tag{1.9}$$

$$G^{ijkm}_{ln\beta\gamma} = G^{jikm}_{nl\beta\gamma}, \qquad G^{ijkm}_{l0\beta\gamma} = G^{jikm}_{l0\beta\gamma}, \tag{1.10}$$

$$G^{ijkm}_{l\alpha p\gamma} = G^{kjim}_{p\alpha l\gamma}, \qquad G^{ijkm}_{l\alpha 0\gamma} = G^{kjim}_{l\alpha 0\gamma}, \tag{1.11}$$

$$P^{ijkm}_{l\beta q} = P^{mjki}_{q\beta l}, \qquad P^{ijkm}_{l\beta 0} = P^{mjki}_{l\beta 0}, \tag{1.12}$$

$$P^{ijkm}_{lp\gamma} = P^{kjim}_{pl\gamma}, \qquad P^{ijkm}_{l0\gamma} = P^{kjim}_{l0\gamma}, \tag{1.13}$$

$$Q^{ijkm}_{lq} = Q^{mjki}_{ql}, \qquad Q^{ijkm}_{l0} = Q^{mjki}_{l0}. \tag{1.14}$$

In order to establish the global existence, we require that the quadratic terms satisfy the following null condition:

$$B^{ijk}_{lmn}\xi_i\xi_j\xi_k\xi_l\xi_m\xi_n = 0, \quad \forall \xi \in S^2. \tag{1.15}$$

1.1. Notations

Let

$$\partial_0 = \frac{\partial}{\partial t}, \qquad \partial_i = \frac{\partial}{\partial x^i}, \tag{1.16}$$

$$\partial = (\partial_0, \nabla) = (\partial_0, \partial_1, \partial_2, \partial_3), \quad \nabla = (\partial_1, \partial_2, \partial_3). \tag{1.17}$$

The angular momentum operators will be written as

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla, \tag{1.18}$$

where $\wedge$ is the usual vector cross product. The spatial derivatives can be decomposed into radial and angular components as follows:

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r}\wedge \Omega, \tag{1.19}$$

in which $r = |x|$, $\partial_r = \frac{x}{r}\cdot\nabla$.

As in [13], we introduce the vectorial modification of the angular momentum operators

$$\widetilde{\Omega}_l = \Omega_l I + U_l \quad (l = 1, 2, 3) \tag{1.20}$$

with

$$U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad U_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.21}$$

which will play an important role in this paper. The scaling operator

$$S = t\partial_t + r\partial_r \tag{1.22}$$

is also important, but, in what follows, we will use

$$\widetilde{S} = S - 1. \tag{1.23}$$

Let $\Gamma = (\Gamma_0, \ldots, \Gamma_7) = (\partial, \widetilde{\Omega}, \widetilde{S})$. It is easy to see that the commutator of any two operators in Γ is either 0 or is in the span of Γ, and, in particular, the commutator of ∇ and Γ in the span of ∇, denoted by

$$[\nabla, \Gamma] = \nabla. \tag{1.24}$$

Moreover, Γ^a, $a = (a_1, \ldots, a_k)$, will denote an ordered product of $k = |a|$ vector fields: $\Gamma^a = \Gamma_{a_1}\cdots\Gamma_{a_k}$.

We will also have to use the operators $D = \sqrt{-\Delta}$ and $|D|^{-1}$ which are easily defined by their symbols: $|\xi|$ and $|\xi|^{-1}$.

Let

$$\Lambda = (\Lambda_1, \ldots, \Lambda_7) = (\nabla, \widetilde{\Omega}, r\partial_r - 1), \tag{1.25}$$

which is the time-independent analog of Γ. Then Λ has the same commutation properties as Γ. Moreover, $D = \sqrt{-\Delta}$ and $|D|^{-1}$ commute with $\Lambda_1, \ldots, \Lambda_6$, while

$$\big[\Lambda_7, |D|\big] = |D| \quad \text{and} \quad \big[\Lambda_7, |D|^{-1}\big] = |D|^{-1}, \tag{1.26}$$

which can be checked by using the Fourier transform. Define

$$H^k_\Lambda = \big\{ f \in L^2\big(\mathbb{R}^3\big)^3\colon\ \Lambda^a f \in \big(L^2\big(\mathbb{R}^3\big)\big)^3,\ |a| \leqslant k \big\} \tag{1.27}$$

with the norm

$$\|f\|^2_{H^k_\Lambda} = \sum_{|a|\leqslant k} \big\|\Lambda^a f\big\|^2_{L^2(\mathbb{R}^3)}. \tag{1.28}$$

By the commutation relation (1.26), it follows that

$$|D| : H^k_\Lambda \to H^{k-1}_\Lambda. \tag{1.29}$$

The energy associated to the linear elastodynamic operator is

$$E_1\big(u(t)\big) = \frac{1}{2}\int_{\mathbb{R}^3} \Big[\big|\partial_t u(t)\big|^2 + c_2^2\big|\nabla u(t)\big|^2 + \big(c_1^2 - c_2^2\big)\big|\nabla\cdot u(t)\big|^2\Big]\,dx, \tag{1.30}$$

and higher energies are defined by

$$E_k\big(u(t)\big) = \sum_{|a|\leqslant k-1} E_1\big(\Gamma^a\big(u(t)\big)\big) \quad (k \geqslant 2). \tag{1.31}$$

In order to control the remaining derivatives of u up to order k, we also introduce nonlocal energies

$$\mathcal{E}_0\big(u(t)\big) = E_1\big(|D|^{-1}u(t)\big) \quad \text{and} \quad \mathcal{E}_k\big(u(t)\big) = \sum_{|a|\leqslant k} \mathcal{E}_0\big(\Gamma^a u(t)\big). \tag{1.32}$$

Thus, we have

$$E_k\big(u(t)\big) \leqslant C\mathcal{E}_k\big(u(t)\big), \qquad \mathcal{E}_{k-1}\big(\nabla\big(u(t)\big)\big) \leqslant CE_k\big(u(t)\big), \tag{1.33}$$

$$\sum_{|a|\leqslant k} \big\|\Gamma^a u\big\|^2_{L^2} \leqslant C\mathcal{E}_k\big(u(t)\big) \quad \text{and} \quad \sum_{|a|\leqslant k-1} \big\|\partial\Gamma^a u(t)\big\|^2_{L^2} \leqslant CE_k\big(u(t)\big). \tag{1.34}$$

The solution will be constructed in the space

$$X^k(T)=\left\{u(t,x)\colon\ u\in\bigcap_{j=0}^{k}C^j\bigl([0,T);H_\wedge^{k-j}\bigr),\ \ \partial_t u\in\bigcap_{j=0}^{k}C^j\bigl([0,T);|D|H_\wedge^{k-j}\bigr)\right\}. \tag{1.35}$$

For $u\in X^k(T)$, the norms $E_k(u(t))$ and $\mathcal{E}_k(u(t))$ are finite for $0\leqslant t<T$.

We will use the following weighted L^2-norms

$$M_k\bigl(u(t)\bigr)=\sum_{\alpha=1}^{2}\sum_{|a|\leqslant k-2}\bigl\|\langle c_\alpha tt-r\rangle P_\alpha\partial\nabla\Gamma^a u(t)\bigr\|_{L^2}, \tag{1.36}$$

$$J_k\bigl(u(t)\bigr)=\sum_{|a|\leqslant k-1}\bigl\|\langle c_\alpha t-r\rangle\partial\Gamma^a u(t)\bigr\|_{L^2} \tag{1.37}$$

with c_α $(\alpha=1,2)$ denote wave speeds, here and hereafter we will use the notation

$$\langle\rho\rangle=\bigl(1+|\rho|^2\bigr)^{\frac12} \tag{1.38}$$

for a scalar or a vector ρ. P_1 and P_2 are defined by the orthogonal projections onto radial and transverse directions respectively, namely

$$P_1u(x)=\frac{x}{r}\otimes\frac{x}{r}u(x)=\frac{x}{r}\left\langle\frac{x}{r},u(x)\right\rangle \tag{1.39}$$

and

$$P_2u(x)=[I-P_1]u(x)=-\frac{x}{r}\wedge\left(\frac{x}{r}\wedge u(x)\right) \tag{1.40}$$

in which $\langle,\rangle$ denotes the inner product in $\mathbb{R}^3$.

1.2. Main result

The main result in this paper is

Theorem 1.1. *Suppose that the symmetry conditions* (1.8)–(1.14) *as well as the null condition* (1.15) *are satisfied and* $k\geqslant 9$. *Then there exists a positive constant* ε *so small that if the initial data* $f\in H_\Lambda^k$ *and* $g\in H_\Lambda^{k-1}$ $(k\geqslant 6)$ *satisfy*

$$\|f\|_{H_\Lambda^k}+\|f\|_{H_\Lambda^{k-1}}+\bigl\||D|^{-1}g\bigr\|_{H_\Lambda^k}<\varepsilon, \tag{1.41}$$

then the Cauchy problem (1.1) *admits a unique global solution* $u=u(t,x)$ *satisfying*

$$\sup_{0\leqslant t<\infty}\bigl[E_k\bigl(u(t)\bigr)+(1+t)^{-\delta_0}\mathcal{E}_k\bigl(u(t)\bigr)\bigr]\leqslant 2\varepsilon \tag{1.42}$$

for some $\delta_0>0$.

This paper is organized as follows: in the next section we give some preliminaries which will be used later. Section 3 will be devoted to weighted L^2 estimates. Making use of the estimates from Sections 2 and 3, we will give the proof of Theorem 1.1 in Section 4.

2. Preliminaries

In this section we collect several lemmas concerning some commutation relations, some estimates of the null forms, and some Sobolev-type inequalities.

We begin with the commutation relations of the vector fields Γ with respect to the operator L:

$$[L,\partial]=[L,\widetilde{\Omega}]=0,\qquad [L,S]=2L. \tag{2.1}$$

In order to establish the energy estimates, we will use the commutation properties of the vector fields Γ with nonlinear terms. For this purpose, we need to verify that the null structure is preserved under differentiation.

Lemma 2.1. *Let* u *be a smooth solution of* (1.1). *Then, for any* Γ^a, *the following equality holds*:

$$L\Gamma^a u(t)=\Gamma^a Lu+\sum_{|b|\leqslant|a|-1}\Gamma^b Lu. \tag{2.2}$$

Proof. (2.2) can be obtained by the method as in T.T. Li and Y.M. Chen [10] for wave operators. □

The next lemma which will play a key role for estimating the energies, gains an additional decay for the nonlinearities with the null structure (1.15).

Lemma 2.2. *Suppose that $u, v, w \in H^2_\wedge$. Suppose furthermore that nonlinear form N satisfies the null condition* (1.15). *Let $\mathcal{N} = \{(\alpha,\beta,\gamma) \neq (1,1,1), (2,2,2)\}$ be the set of nonresonant indices. Then*

$$\begin{aligned}\big|\langle u(x), N\big(v(x), w(x)\big)\rangle\big| \leqslant{}& \frac{C}{r}\big|u(x)\big| \sum_{|a|\leqslant 1}\big[\big|\nabla\widetilde{\Omega}^a v(x)\big|\big|\nabla w(x)\big| + \big|\nabla\widetilde{\Omega}^a w(x)\big|\big|\nabla v(x)\big| \\ &+ \big|\nabla^2 v(x)\big|\big|\widetilde{\Omega}^a w(x)\big| + \big|\nabla^2 w(x)\widetilde{\Omega}^a v(x)\big|\big] \\ &+ C\sum_{\mathcal{N}}\big|P_\alpha u(x)\big|\big[\big|P_\beta\nabla^2 v(x)\big|\big|P_\gamma\nabla w(x)\big| + \big|P_\beta\nabla^2 w(x)\big|\big|P_\gamma\nabla v(x)\big|\big],\end{aligned} \tag{2.3}$$

here and hereafter $\alpha,\beta,\gamma = 1,2$, C denotes a positive constant.

Proof. The proof of (2.3) can be found in T.C. Sideris [13]. □

Lemma 2.3. *For any given $u \in C_c^\infty(\mathbb{R}^3)$, $a \geqslant 0$ we have*

$$\langle r\rangle^{\frac{1}{2}}\big|\Gamma^a u(t)\big| \leqslant C E^{\frac{1}{2}}_{|a|+2}\big(u(t)\big), \tag{2.4a}$$

$$\langle r\rangle\big|\Gamma^a\big(u(t)\big)\big| \leqslant C E^{\frac{1}{2}}_{|a|+2}\big(u(t)\big) + C\mathcal{E}^{\frac{1}{2}}_{|a|+2}\big(u(t)\big), \tag{2.4b}$$

$$\langle r\rangle\big|\partial\Gamma^a\big(u(t)\big)\big| \leqslant C E^{\frac{1}{2}}_{|a|+3}\big(u(t)\big), \tag{2.4c}$$

$$\langle r\rangle\langle c_\alpha t - r\rangle\big|\partial\Gamma^a u\big| \leqslant C M_{|a|+3}\big(u(t)\big) + C J_{|a|+3}\big(u(t)\big), \tag{2.4d}$$

$$\langle r\rangle\langle c_\alpha t - r\rangle\big|P_\alpha\partial\nabla\Gamma^a\big(u(t,x)\big)\big| \leqslant C M_{|a|+4}\big(u(t)\big), \tag{2.4e}$$

$$\langle r\rangle^{\frac{1}{2}}\langle c_\alpha t - r\rangle\big|\Gamma^a u\big| \leqslant C M_{|a|+2}\big(u(t)\big) + C\mathcal{E}^{\frac{1}{2}}_{|a|+1}\big(u(t)\big) + C J_{|a|+3}\big(u(t)\big), \tag{2.4f}$$

$$\langle r\rangle^{\frac{1}{2}}\langle c_\alpha t - r\rangle\big|\partial\Gamma^a u\big| \leqslant C E^{\frac{1}{2}}_{|a|+2}\big(u(t)\big) + C M_{|a|+3}\big(u(t)\big), \tag{2.4g}$$

provided that the norms on the right-hand side are finite.

Proof. (2.4a), (2.4c) and (2.4e) can be found in Proposition 3.3 of T.C. Sideris [13]. The proof of (2.4g) is given by Lemma 4.3 in K. Hidano [6]. We need only to prove (2.4b), (2.4d) and (2.4f).

When $r > 1$, we can get (2.4b) from

$$\begin{aligned} r\big|\Gamma^a\big(u(t)\big)\big| &\leqslant C\sum_{|b|\leqslant 1}\big\|\partial_r\widetilde{\Omega}^b\Gamma^a\big(u(t)\big)\big\|_{L^2} + C\sum_{|b|\leqslant 2}\big\|\widetilde{\Omega}^b\Gamma^a u\big\|_{L^2(|y|\geqslant r)} \\ &\leqslant C E^{\frac{1}{2}}_{|a|+2}\big(u(t)\big) + C\mathcal{E}^{\frac{1}{2}}_{|a|+2}\big(u(t)\big). \end{aligned}$$

Moreover, by Sobolev embedding theorem, (2.4b) also holds for $r < 1$:

$$\big|\Gamma^a u(t)\big| \leqslant \sum_{|b|=1}^{2}\big\|\nabla^b\Gamma^a u(t)\big\|_{L^2} \leqslant C E^{\frac{1}{2}}_{|a|+2}\big(u(t)\big).$$

For $r > 1$, we have

$$\begin{aligned} r\langle c_\alpha t - r\rangle\big|\partial\Gamma^a u(x)\big| &\leqslant \sum_{|b|\leqslant 1}\big\|\langle c_\alpha t - \rho\rangle\partial_r\widetilde{\Omega}^b\partial\Gamma^a u\big\|_{L^2(|y|\geqslant r)} + C\sum_{|b|\leqslant 2}\big\|\langle c_\alpha t - \rho\rangle\widetilde{\Omega}^b\partial\Gamma^a u\big\|_{L^2(|y|\geqslant r)} \\ &\leqslant C\sum_\alpha\sum_{|b|\leqslant 1}\big\|\langle c_\alpha t - \rho\rangle P_\alpha\partial_r\widetilde{\Omega}^b\partial\Gamma^a u\big\|_{L^2(|y|\geqslant r)} + C\sum_{|b|\leqslant 2}\big\|\langle c_\alpha t - \rho\rangle\widetilde{\Omega}^b\partial\Gamma^a u\big\|_{L^2(|y|\geqslant r)} \\ &\leqslant C M_{|a|+3}\big(u(t)\big) + C J_{|a|+3}\big(u(t)\big). \end{aligned}$$

On the other hand, for $r \leqslant 1$, let Φ be a smooth, compactly supported function such that

$$\Phi(x) = \begin{cases} 1, & \text{for } |x| \leqslant 1, \\ 0, & \text{for } |x| \geqslant 2. \end{cases}$$

It follows from the Sobolev embedding theorem that

$$\begin{aligned}\left|\langle c_\alpha t-r\rangle\partial\Gamma^a u(t,x)\right| &\leqslant C(1+t)\left|\Phi\partial\Gamma^a u\right|\\ &\leqslant C(1+t)\sum_{|b|=1}^{2}\left\|\nabla^b\left(\Phi\partial\Gamma^a u\right)\right\|_{L^2}\\ &\leqslant C(1+t)\sum_{|b|=1}^{2}\left\|\nabla^b\partial\Gamma^a u\right\|_{L^2(|x|\leqslant 2)}+C(1+t)\sup_{1<|x|<2}\left|\partial\Gamma^a u\right|\\ &\leqslant CM_{|a|+3}\big(u(t)\big)+CJ_{|a|+3}\big(u(t)\big).\end{aligned}$$

The proof of (2.4f) starts with the following radius-angular mixed-norm inequality:

$$r^{\frac{1}{2}}\left(\int_{S^2}\left|v(r\omega)\right|^4 d\omega\right)^{\frac{1}{4}}\leqslant C\|\nabla v\|_{L^2}. \tag{2.5}$$

For $r>1$, it follows from (2.5) and the Sobolev embedding theorem on S^2 that

$$\begin{aligned}r^{\frac{1}{2}}\left|\langle c_\alpha t-r\rangle\Gamma^a u\right| &\leqslant C\sum_{|b|\leqslant 1}\left(\int_{S^2}\widetilde{\Omega}^b\langle c_\alpha t-r\rangle\left|\Gamma^a u\right|^4 d\omega\right)^{\frac{1}{4}}\\ &\leqslant C\sum_{|b|\leqslant 1}\left\|\langle c_\alpha t-r\rangle\widetilde{\Omega}^b\Gamma^a u\right\|_{H^1}\\ &\leqslant C\sum_{|b|\leqslant|a|+1}\left\|\Gamma^b u\right\|_{L^2}+C\sum_{|b|\leqslant|a|+1}\left\|\langle c_\alpha t-r\rangle\nabla\Gamma^b u\right\|_{L^2}\\ &\leqslant C\mathcal{E}_{|a|+1}^{\frac{1}{2}}\big(u(t)\big)+CJ_{|a|+2}\big(u(t)\big).\end{aligned}$$

Moreover, for $r<1$. We have

$$\begin{aligned}\left|\langle c_\alpha t-r\rangle\Gamma^a u\right| &\leqslant C(1+t)\left|\Phi\Gamma^a u\right|\leqslant C(1+t)\sum_{|b|=1}^{2}\left\|\nabla^b\left(\Phi\Gamma^a u\right)\right\|_{L^2}\\ &\leqslant C(1+t)\sum_{|b|=1}^{2}\left\|\nabla^b\Gamma^a u\right\|_{L^2(|x|\leqslant 2)}+C\sup_{1<|x|<2}(1+t)\left|\Gamma^a u\right|+C\sup_{1<|x|<2}\left|\langle c_\alpha t-r\rangle\Gamma^a u\right|\\ &\leqslant C\left[M_{|a|+2}\big(u(t)\big)+\mathcal{E}_{|a|+1}^{\frac{1}{2}}\big(u(t)\big)+J_{|a|+2}\big(u(t)\big)\right]. \qquad\Box\end{aligned}$$

3. Weighted L^2-estimates

It is necessary to control the weighted L^2-norm $M_k(u(t))$ and $J_k(u(t))$ by $E_k^{\frac{1}{2}}(u(t))$ and $\mathcal{E}_k^{\frac{1}{2}}(u(t))$ respectively for the completion of the energy integral argument.

Lemma 3.1. *Let $u\in X^k(T)$ be a solution of* (1.1)*. Then*

$$\sum_{|a|\leqslant k-2}\sum_{\alpha=1}^{2}\left\|\langle c_\alpha t-r\rangle P_\alpha\partial\nabla\Gamma^a u(t,\cdot)\right\|_{L^2}\leqslant CE_k^{\frac{1}{2}}\big(u(t)\big)+Ct\sum_{|a|\leqslant k-2}\left\|L\Gamma^a u\right\|_{L^2}, \tag{3.1}$$

$$\sum_{|a|\leqslant k-1}\left\|\langle c_\alpha t-r\rangle\partial\Gamma^a u\right\|_{L^2}\leqslant C\left[\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big)+t\sum_{|a|\leqslant k-1}\left\||D|^{-1}L\Gamma^a u\right\|_{L^2}\right]. \tag{3.2}$$

Proof. By Lemma 3.4 in T.C. Sideris [13], we get (3.1). On the other hand, (3.2) can be obtained as follows:

$$\begin{aligned}\left\|\langle c_\alpha t-r\rangle\partial\Gamma^a u\right\|_{L^2} &=\left\|\langle c_\alpha t-r\rangle\partial(-\triangle)(-\triangle)^{-1}\Gamma^a u\right\|_{L^2}\\ &=\left\|\langle c_\alpha t-r\rangle\partial\partial_p\big(\partial_p|D|^{-1}\big)|D|^{-1}\Gamma^a u\right\|_{L^2}\leqslant CE_1\big(|D|^{-1}\Gamma^a u\big)+t\left\||D|^{-1}L\Gamma^a u\right\|_{L^2}\\ &\leqslant\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big)+t\left\||D|^{-1}L\Gamma^a u\right\|_{L^2}. \qquad\Box\end{aligned}$$

Lemma 3.2. *Let u be a smooth solution of (1.1). Assume $k \geqslant 5$, then for any a with $|a| \leqslant k-2$, we have*

$$t\|L\Gamma^a u\|_{L^2} \leqslant C\big[E_k^{\frac{1}{2}}\big(u(t)\big)M_k\big(u(t)\big) + E_k^{\frac{3}{2}}\big(u(t)\big) + E_k\big(u(t)\big)M_k\big(u(t)\big)\big] \tag{3.3}$$

and for any a with $|a| \leqslant k-1$, we have

$$\begin{aligned} t\||D|^{-1}L\Gamma^a u\|_{L^2} \leqslant C\big[&E_k^{\frac{1}{2}}\big(u(t)\big)J_k\big(u(t)\big) + E_k\big(u(t)\big)J_k\big(u(t)\big) + E_k^{\frac{3}{2}}\big(u(t)\big) \\ &+ E_k\big(u(t)\big)M_k\big(u(t)\big) + E_k\big(u(t)\big)\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big)\big]. \end{aligned} \tag{3.4}$$

Proof. From Lemma 2.1 we get

$$\begin{aligned} t\|L\Gamma^a u\|_{L^2} \leqslant Ct\big\{&\|\nabla^2\Gamma^b u\nabla\Gamma^c u\|_{L^2} + \|\Gamma^b u\Gamma^c u\nabla\Gamma^d u\|_{L^2} + \|\Gamma^b u\nabla\Gamma^c u\partial\Gamma^d u\|_{L^2} \\ &+ \|\Gamma^b u\Gamma^c u\partial\nabla\Gamma^d u\|_{L^2} + \|\nabla\Gamma^b u\partial\Gamma^c u\partial\Gamma^d u\|_{L^2} \\ &+ \|\Gamma^b u\partial\Gamma^c u\partial\nabla\Gamma^d u\|_{L^2} + \|\partial\Gamma^b u\partial\Gamma^c u\partial\nabla\Gamma^d u\|_{L^2}\big\}. \end{aligned} \tag{3.5}$$

For the estimates of the first group on the right-hand side of (3.5), see Lemma 3.5 in T.C. Sideris [13]. We need only to prove other groups. Since $b+c+d \leqslant k-2$, using (2.4a) and Hardy inequality, we have

$$\begin{aligned} &\|\Gamma^b u\Gamma^c u\nabla\Gamma^d u\|_{L^2} \\ &\leqslant C\langle t\rangle^{-1}\big[\|\langle r\rangle\Gamma^b u\Gamma^c u\nabla\Gamma^d u\|_{L^2(r\geqslant\frac{\langle c_2 t\rangle}{2})} + \|\langle c_\alpha t - r\rangle\Gamma^b u\Gamma^c u\nabla\Gamma^d u\|_{L^2(r<\frac{\langle c_2 t\rangle}{2})}\big] \\ &\leqslant C\langle t\rangle^{-1}\left[\|\langle r\rangle^{\frac{1}{2}}\Gamma^b u\|_{L^\infty}\|\langle r\rangle^{\frac{1}{2}}\Gamma^c u\|_{L^\infty}\|\nabla\Gamma^d u\|_{L^2} + \|\langle r\rangle^{\frac{1}{2}}\Gamma^b u\|_{L^\infty}\|\langle r\rangle^{\frac{1}{2}}\Gamma^c u\|_{L^\infty}\left\|\frac{1}{r}\langle c_\alpha t - r\rangle\nabla\Gamma^d u\right\|_{L^2}\right] \\ &\leqslant C\langle t\rangle^{-1}E_k\big(u(t)\big)\big[E_k^{\frac{1}{2}}\big(u(t)\big) + M_k\big(u(t)\big)\big]. \end{aligned}$$

Similarly we get

$$\begin{aligned} &\|\Gamma^b u\nabla\Gamma^c u\partial\Gamma^d u\|_{L^2} \leqslant C\langle t\rangle^{-1}E_k\big(u(t)\big)\big[E_k^{\frac{1}{2}}\big(u(t)\big) + M_k\big(u(t)\big)\big], \\ &\|\Gamma^b u\Gamma^c u\partial\nabla\Gamma^d u\|_{L^2} \leqslant C\langle t\rangle^{-1}E_k\big(u(t)\big)\big[E_k^{\frac{1}{2}}\big(u(t)\big) + M_k\big(u(t)\big)\big], \\ &\|\nabla\Gamma^b u\partial\Gamma^c u\partial\Gamma^d u\|_{L^2} \leqslant C\langle t\rangle^{-1}E_k\big(u(t)\big)\big[E_k^{\frac{1}{2}}\big(u(t)\big) + M_k\big(u(t)\big)\big], \\ &\|\Gamma^b u\partial\Gamma^c u\partial\nabla\Gamma^d u\|_{L^2} \leqslant C\langle t\rangle^{-1}E_k\big(u(t)\big)M_k\big(u(t)\big) \end{aligned}$$

and

$$\|\partial\Gamma^b u\partial\Gamma^c u\partial\nabla\Gamma^d u\|_{L^2} \leqslant C\langle t\rangle^{-1}E_k\big(u(t)\big)M_k\big(u(t)\big).$$

This completes the proof of the (3.3).

By Lemma 2.1, we know

$$\begin{aligned} t\||D|^{-1}L\Gamma^a u\|_{L^2} \leqslant Ct\big\{&\|\nabla\Gamma^b u\nabla\Gamma^c u\|_{L^2} + \|\Gamma^b u\Gamma^c u\Gamma^d u\|_{L^2} + \|\Gamma^b u\Gamma^c u\partial\Gamma^d u\|_{L^2} \\ &+ \|\Gamma^b u\partial\Gamma^c u\partial\Gamma^d u\|_{L^2} + \|\partial\Gamma^b u\partial\Gamma^c u\partial\Gamma^d u\|_{L^2}\big\}. \end{aligned} \tag{3.6}$$

For the first group on the right-hand side of (3.6), by (2.4c) and the definition of $J_k(u(t))$, we have

$$\begin{aligned} \|\nabla\Gamma^b u\nabla\Gamma^c u\|_{L^2} &\leqslant C\langle t\rangle^{-1}\|\langle r\rangle\langle c_\alpha t - r\rangle\nabla\Gamma^b u\nabla\Gamma^c u\|_{L^2} \\ &\leqslant C\langle t\rangle^{-1}\|\langle r\rangle\nabla\Gamma^b u\|_{L^\infty}\|\langle c_\alpha t - r\rangle\nabla\Gamma^c u\|_{L^2} \quad (b+3 \leqslant k) \\ &\leqslant C\langle t\rangle^{-1}E_k^{\frac{1}{2}}\big(u(t)\big)J_k\big(u(t)\big). \end{aligned}$$

Since $b+c+d \leqslant k-1$, without loss of generality, we may assume that $b+2 \leqslant k$ and $c+2 \leqslant k$. By (2.4a) and Hardy inequality, we have

$$\begin{aligned} \|\Gamma^b u\Gamma^c u\Gamma^d u\|_{L^2} &\leqslant C\langle t\rangle^{-1}\big[\|\langle r\rangle\Gamma^b u\Gamma^c u\Gamma^d u\|_{L^2(r\geqslant\frac{\langle c_2 t\rangle}{2})} + \|\langle c_\alpha t - r\rangle\Gamma^b u\Gamma^c u\Gamma^d u\|_{L^2(r<\frac{\langle c_2 t\rangle}{2})}\big] \\ &\leqslant C\langle t\rangle^{-1}\left[\|\langle r\rangle^{\frac{1}{2}}\Gamma^b u\|_{L^\infty}\|\langle r\rangle^{\frac{1}{2}}\Gamma^c u\|_{L^\infty}\|\Gamma^d u\|_{L^2} + \|\langle r\rangle^{\frac{1}{2}}\Gamma^b u\|_{L^\infty}\|\langle r\rangle^{\frac{1}{2}}\Gamma^c u\|_{L^\infty}\left\|\frac{1}{r}\langle c_\alpha t - r\rangle\Gamma^d u\right\|_{L^2}\right] \\ &\leqslant C\langle t\rangle^{-1}\big[E_k\big(u(t)\big)\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + E_k\big(u(t)\big)\big(\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + \|\langle c_\alpha t - r\rangle\nabla\Gamma^d u\|_{L^2}\big)\big] \\ &\leqslant \langle t\rangle^{-1}E_k\big(u(t)\big)\big[\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + J_k\big(u(t)\big)\big]. \end{aligned}$$

Similarly, we have

$$\left\|\Gamma^b u \Gamma^c u \partial \Gamma^d u\right\|_{L^2} \leqslant C\langle t\rangle^{-1} E_k\big(u(t)\big) J_k\big(u(t)\big),$$

$$\left\|\Gamma^b u \partial \Gamma^c u \partial \Gamma^d u\right\|_{L^2} \leqslant C\langle t\rangle^{-1} E_k\big(u(t)\big)\big[E_k^{\frac{1}{2}}\big(u(t)\big) + M_k\big(u(t)\big)\big]$$

and

$$\left\|\partial \Gamma^b u \partial \Gamma^c u \partial \Gamma^d u\right\|_{L^2} \leqslant C\langle t\rangle^{-1} E_k\big(u(t)\big) J_k\big(u(t)\big). \qquad \square$$

Lemma 3.3. *Let $u \in X^k(T)$ be a solution of (1.1). Suppose that $k \geqslant 5$ and $\sup_{0\leqslant t<T} E_k^{\frac{1}{2}}(u(t)) < \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small. Then*

$$M_k\big(u(t)\big) \leqslant C E_k^{\frac{1}{2}}\big(u(t)\big), \tag{3.7}$$

$$J_k\big(u(t)\big) \leqslant C \mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big). \tag{3.8}$$

Proof. By Lemmas 3.1 and 3.2, when $\varepsilon_0 > 0$ is small enough, we have

$$\begin{aligned} M_k\big(u(t)\big) &\leqslant C\big[E_k^{\frac{1}{2}}\big(u(t)\big) + E_k^{\frac{1}{2}}\big(u(t)\big)M_k\big(u(t)\big) + E_k^{\frac{3}{2}}\big(u(t)\big) + E_k\big(u(t)\big)M_k\big(u(t)\big)\big] \\ &\leqslant C\big[E_k^{\frac{1}{2}}\big(u(t)\big) + \varepsilon_0 M_k\big(u(t)\big) + \varepsilon_0^{\frac{3}{2}} + \varepsilon_0^2 M_k\big(u(t)\big)\big] \\ &\leqslant C E_k^{\frac{1}{2}}\big(u(t)\big) \end{aligned}$$

and

$$\begin{aligned} J_k\big(u(t)\big) &\leqslant C\big[\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + E_k^{\frac{1}{2}}\big(u(t)\big)J_k\big(u(t)\big) + E_k\big(u(t)\big)J_k\big(u(t)\big) + E_k^{\frac{3}{2}}\big(u(t)\big) \\ &\quad + E_k\big(u(t)\big)\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + E_k\big(u(t)\big)M_k\big(u(t)\big)\big] \\ &\leqslant C\big[\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + \varepsilon_0 J_k\big(u(t)\big) + \varepsilon_0^2 J_k\big(u(t)\big) + \varepsilon_0^{\frac{3}{2}} + \varepsilon_0^2 \mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big)\big] \\ &\leqslant C\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big)(1+\varepsilon_0) \\ &\leqslant C\mathcal{E}_k^{\frac{1}{2}}. \qquad \square \end{aligned}$$

4. Energy estimates

We now come to the main part of the proof, the derivation of a priori energy estimates for small solutions. Let $u \in X^k(T)$, $k \geqslant 6$, have small initial data. The strategy is to show that

$$\begin{cases} \dfrac{d}{dt} E_k\big(u(t)\big) \leqslant C\langle t\rangle^{-\frac{3}{2}} E_k\big(u(t)\big)\big(\mathcal{E}_k^{\frac{1}{2}}\big(u(t)\big) + \mathcal{E}_k\big(u(t)\big)\big), \\ \dfrac{d}{dt} \mathcal{E}_k\big(u(t)\big) \leqslant C\langle t\rangle^{-1} \mathcal{E}_k\big(u(t)\big)\big(E_k^{\frac{1}{2}}\big(u(t)\big) + E_k\big(u(t)\big)\big). \end{cases} \tag{4.1}$$

Thus, for sufficiently small initial values $E_k(u(0))$ and $\mathcal{E}_k(u(0))$, there exists a $\delta > 0$ such that

$$E_k\big(u(t)\big) + (1+t)^{-\delta}\mathcal{E}_k\big(u(t)\big) \leqslant 2\big[E_k\big(u(0)\big) + \mathcal{E}_k\big(u(0)\big)\big]$$

for all $t \geqslant 0$.

To obtain the first inequality of (4.1), the nonlinearity will be estimated in a different way for $r \leqslant \frac{\langle C_2 t\rangle}{2}$ and $r \geqslant \frac{\langle C_2 t\rangle}{2}$ respectively. In the first zone, the $\langle t\rangle^{-\frac{3}{2}}$ decaying factor comes from the weighted L^2 and L^∞ estimates. While for large r, the weighted L^∞ estimate and the null condition are used. The inequality for the nonlocal energy depends on the divergence form of the nonlinearity and the weighted L^2 estimate.

For any given a with $|a| \leqslant k-1$, taking the product of $L\Gamma^a u$ with $\partial_t \Gamma^a u$, integrating on $\mathbb{R}^3$, and then summing up with respect to a, we get

$$\begin{aligned} E_k'\big(u(t)\big) &= \sum_{|a|\leqslant k-1} \int_{\mathbb{R}^3} \big\langle \partial_t \Gamma^a u, L\Gamma^a u\big\rangle dx \\ &= \sum_{|a|\leqslant k-1} \int_{\mathbb{R}^3} \big\langle \partial_t \Gamma^a u, \Gamma^a L u\big\rangle dx + \sum_{|a|\leqslant k-1} \int_{\mathbb{R}^3} \big\langle \partial_t \Gamma^a u, \big[L, \Gamma^a\big]u\big\rangle dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{|a|=k-1} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i B^{ijk}_{lmn} \partial_l\big(\partial_m \Gamma^a u^j \partial_n u^k\big)\,dx \\
&\quad + \sum_{|a|=k-1} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i G^{ijkm}_{l\alpha\beta\gamma} \partial_l\big(\partial_\alpha u^j \partial_\beta u^k \partial_\gamma \Gamma^a u^m\big)\,dx \\
&\quad + \sum_{|a|=k-1} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i P^{ijkm}_{l\beta\gamma} \partial_l\big(u^j \partial_\beta u^k \partial_\gamma \Gamma^a u^m\big)\,dx \\
&\quad + \sum_{|a|=k-1} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i Q^{ijkm}_{l\gamma} \partial_l\big(u^j u^k \partial_\gamma \Gamma^a u^m\big)\,dx \\
&\quad + \sum_{\substack{|a|\leqslant k-1\\ b+c=a,\, b\neq a}} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i B^{ijk}_{lmn} \partial_l\big(\partial_m \Gamma^b u^j \partial_n \Gamma^c u^k\big)\,dx \\
&\quad + \sum_{\substack{|a|\leqslant k-1\\ b+c+d=a,\, d\neq a}} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i G^{ijkm}_{l\alpha\beta\gamma} \partial_l\big(\partial_\alpha \Gamma^b u^j \partial_\beta \Gamma^c u^k \partial_\gamma \Gamma^d u^m\big)\,dx \\
&\quad + \sum_{\substack{|a|\leqslant k-1\\ b+c+d=a,\, d\neq a}} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i P^{ijkm}_{l\beta\gamma} \partial_l\big(\Gamma^b u^j \partial_\beta \Gamma^c u^k \partial_\gamma \Gamma^d u^m\big)\,dx \\
&\quad + \sum_{\substack{|a|\leqslant k-1\\ b+c+d=a,\, d\neq a}} \int_{\mathbb{R}^3} \partial_t \Gamma^a u^i Q^{ijkm}_{l\gamma} \partial_l\big(\Gamma^b u^j \Gamma^c u^k \partial_\gamma \Gamma^d u^m\big)\,dx \\
&\quad + \sum_{|a|\leqslant k-1} \int_{\mathbb{R}^3} \partial_t \Gamma^a u \Gamma^a H(u,u,u)\,dx + \sum_{|a|\leqslant k-1} \int_{\mathbb{R}^3} \big\langle \partial_t \Gamma^a u, \big[L, \Gamma^a\big]u\big]\big\rangle\,dx. \qquad (4.2)
\end{aligned}
$$

Integrating by parts and using the symmetry conditions (1.8)–(1.14), the modified energy can be expressed as

$$
\begin{aligned}
\widetilde{E}_k\big(u(t)\big) &= E_k\big(u(t)\big) + \int_{\mathbb{R}^3} B^{ijk}_{lmn} \partial_l \Gamma^a u^i \partial_m \Gamma^a u^j \partial_n u^k\,dx \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} G^{ijkm}_{l\alpha\beta q} \partial_\alpha u^j \partial_\beta u^k \partial_l \Gamma^a u^i \partial_q \Gamma^a u^m\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} G^{ijkm}_{ln\beta\gamma} \partial_l \Gamma^a u^i \partial_n \Gamma^a u^j \partial_\beta u^k \partial_\gamma u^m\,dx \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} G^{ijkm}_{l\alpha p\gamma} \partial_l \Gamma^a u^i \partial_p \Gamma^a u^k \partial_\alpha u^j \partial_\gamma u^m\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} P^{ijkm}_{l\beta q} u^j \partial_\beta u^k \partial_l \Gamma^a u^i \partial_q \Gamma^a u^m\,dx \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} P^{ijkm}_{lp\gamma} u^j \partial_p \Gamma^a u^k \partial_l \Gamma^a u^i \partial_\gamma u^m\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} Q^{ijkm}_{lq} u^j u^k \partial_l \Gamma^a u^i \partial_q \Gamma^a u^m\,dx. \qquad (4.3)
\end{aligned}
$$

Noting that for small solution, we have

$$
cE_k\big(u(t)\big) \leqslant \widetilde{E}_k\big(u(t)\big) \leqslant CE_k\big(u(t)\big), \qquad (4.4)
$$

it is easy to see that

$$
\begin{aligned}
\frac{d}{dt}\widetilde{E}_k\big(u(t)\big) &\leqslant C \sum_{|a|\leqslant k-1} \sum_{\substack{b+c=a\\ b\neq a}} \big\|\partial_t \Gamma^a u\big\|_{L^2} \big\|\nabla^2 \Gamma^b u \nabla \Gamma^c u\big\|_{L^2} \\
&\quad + C \sum_{|a|\leqslant k-1} \sum_{\substack{b+c+d=a\\ d\neq a}} \big\{\big\|\partial_t \Gamma^a u\big\|_{L^2}\big[\big\|\Gamma^b u \Gamma^c u \partial\nabla \Gamma^d u\big\|_{L^2} + \big\|\Gamma^b u \partial \Gamma^c u \partial\nabla \Gamma^d u\big\|_{L^2} + \big\|\partial \Gamma^b u \partial \Gamma^c u \partial\nabla \Gamma^d u\big\|_{L^2}\big]\big\} \\
&\quad + C \sum_{|a|\leqslant k-1} \sum_{b+c+d=a} \big\{\big\|\partial_t \Gamma^a u\big\|_{L^2}\big[\big\|\Gamma^b u \Gamma^c u \partial \Gamma^d u\big\|_{L^2} + \big\|\Gamma^b u \partial \Gamma^c u \partial \Gamma^d u\big\|_{L^2} + \big\|\partial \Gamma^b u \partial \Gamma^c u \partial \Gamma^d u\big\|_{L^2}\big]\big\}.
\end{aligned}
\qquad (4.5)
$$

Treating (4.5) in the same manner as in Lemma 3.2, when $r \leqslant \frac{\langle C_2 t\rangle}{2}$, by Lemma 3.3 we get

$$
\frac{d}{dt}\widetilde{E}_k\big(u(t)\big) \leqslant C\langle t\rangle^{-\frac{3}{2}} \widetilde{E}_k\big(u(t)\big)\big(\widetilde{\mathcal{E}}_k^{\frac{1}{2}}\big(u(t)\big) + \widetilde{\mathcal{E}}_k\big(u(t)\big)\big). \qquad (4.6)
$$

While, when $r > \frac{\langle c_2 t\rangle}{2}$, by (2.3) we get

$$\int_{\mathbb{R}^3} \langle \partial_t \Gamma^a u, N(\Gamma^b u, \Gamma^c u)\rangle dx \leqslant C\|\partial \Gamma^a u\|_{L^2}\left[\left\|\frac{1}{r}\nabla \Gamma^{b+1} u \nabla \Gamma^c u\right\|_{L^2} + \left\|\frac{1}{r}\nabla^2 \Gamma^b u \Gamma^{c+1} u\right\|_{L^2}\right] + \sum_{\mathcal{N}} \int_{\mathbb{R}^3} |P_\alpha \partial_t \Gamma^a u\|P_\beta \nabla^2 \Gamma^b u\|P_\gamma \nabla \Gamma^c u| \, dx. \tag{4.7}$$

Similarly to the method in Lemma 3.2, we obtain

$$\left\|\frac{1}{r}\nabla \Gamma^{b+1} u \nabla \Gamma^c u\right\|_{L^2} + \left\|\frac{1}{r}\nabla^2 \Gamma^b u \Gamma^{c+1} u\right\|_{L^2} \leqslant C\langle t\rangle^{-2} E_k^{\frac{1}{2}}(u(t))\mathcal{E}_k^{\frac{1}{2}}(u(t)).$$

As to the second term on the right-hand side of (4.7), without loss of generality, we may assume $\beta \neq \gamma$, then the inequality $1 \leqslant C\langle t\rangle^{-\frac{3}{2}}\langle r\rangle\langle c_\beta t - r\rangle\langle c_\gamma t - r\rangle^{\frac{1}{2}}$ holds, thus we get

$$\begin{aligned}
&\sum_{\mathcal{N}} \int_{\mathbb{R}^3} |P_\alpha \partial_t \Gamma^a u\|P_\beta \nabla^2 \Gamma^b u\|P_\gamma \nabla \Gamma^c u| \, dx \\
&\leqslant C\langle t\rangle^{-\frac{3}{2}} \left\|\langle r\rangle\langle c_\beta t - r\rangle\langle c_\gamma t - r\rangle^{\frac{1}{2}} P_\alpha \partial_t \Gamma^a u P_\beta \nabla^2 \Gamma^b u P_\gamma \nabla \Gamma^c u\right\|_{L^1} \\
&\leqslant C\langle t\rangle^{-\frac{3}{2}} \begin{cases} \sum_{\beta=1}^{2} \|\partial_t \Gamma^a u\|_{L^2} \|\langle r\rangle\langle c_\beta t - r\rangle P_\beta \nabla^2 \Gamma^b u\|_{L^\infty} \|\langle c_\gamma t - r\rangle P_\gamma \nabla \Gamma^c u\|_{L^2}, & b+4 \leqslant k, \\ \sum_{\beta=1}^{2} \|\partial_t \Gamma^a u\|_{L^2} \|\langle c_\beta t - r\rangle P_\beta \nabla^2 \Gamma^b u\|_{L^2} \|\langle r\rangle\langle c_\gamma t - r\rangle P_\gamma \nabla \Gamma^c u\|_{L^\infty}, & c+3 \leqslant k \end{cases} \\
&\leqslant C\langle t\rangle^{-\frac{3}{2}} E_k^{\frac{1}{2}}(u(t))\mathcal{E}_k(u(t)).
\end{aligned}$$

We may deal with other terms in (4.5) in the same way as in Lemma 3.2. Hence, we get

$$\frac{d}{dt}\widetilde{E}_k(u(t)) \leqslant C\langle t\rangle^{-\frac{3}{2}} \widetilde{E}_k(u(t))\left(\widetilde{\mathcal{E}}_k^{\frac{1}{2}}(u(t)) + \widetilde{\mathcal{E}}_k(u(t))\right). \tag{4.8}$$

The final step is to estimate the nonlocal energy. We apply $|D|^{-1}\Gamma^a u$ to Eq. (1.1), multiply $\partial_t |D|^{-1}\Gamma^a u$, integrate on $\mathbb{R}^3$, and sum up over $|a| \leqslant k$ to get

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_k(u(t)) &= \sum_{|a|\leqslant k} \int_{\mathbb{R}^3} \langle \partial_t |D|^{-1}\Gamma^a u, |D|^{-1} L \Gamma^a u\rangle dx \\
&\leqslant C\|\partial_t |D|^{-1}\Gamma^a u\|_{L^2} \||D|^{-1} L \Gamma^a u\|_{L^2}.
\end{aligned} \tag{4.9}$$

In a similar way to that for $E_k(u(t))$, we get

$$\begin{aligned}
\widetilde{\mathcal{E}}_k(u(t)) = \mathcal{E}_k(u(t)) &- \int_{\mathbb{R}^3} B_{lmn}^{ijk} \partial_l \partial_p (-\triangle)^{-1}\Gamma^a u^i(t) \partial_m \partial_p (-\triangle)^{-1}\Gamma^a u^j \partial_n u^k \, dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} \partial_l \partial_r (-\Delta)^{-1}\Gamma^a u^i(t) G_{l\alpha\beta q}^{ijkm} \partial_\alpha u^j \partial_\beta u^k \partial_q \partial_r (-\Delta)^{-1}\Gamma^a u^m \, dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} G_{ln\beta\gamma}^{ijkm} \partial_l \partial_r (-\Delta)^{-1}\Gamma^a u^i \partial_n \partial_r (-\Delta)^{-1}\Gamma^a u^j \partial_\beta u^k \partial_\gamma u^m \, dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} G_{l\alpha p\gamma}^{ijkm} \partial_l \partial_r (-\Delta)^{-1}\Gamma^a u^i \partial_p \partial_r (-\Delta)^{-1}\Gamma^a u^k \partial_\alpha u^j \partial_\gamma u^m \, dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} P_{l\beta q}^{ijkm} \partial_l \partial_r (-\Delta)^{-1}\Gamma_a u^i \partial_q \partial_r (-\Delta)^{-1}\Gamma_a u^m u^j \partial_\beta u^k \, dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} P_{lp\gamma}^{ijkm} \partial_l \partial_r (-\Delta)^{-1}\Gamma_a u^i \partial_p \partial_r (-\Delta)^{-1}\Gamma_a u^k u^j \partial_\gamma u^m \, dx \\
&- \frac{1}{2}\int_{\mathbb{R}^3} Q_{lq}^{ijkm} \partial_l \partial_r (-\Delta)^{-1}\Gamma^a u^i \partial_q \partial_r (-\Delta)^{-1}\Gamma^a u^m u^j u^k \, dx
\end{aligned}$$

and

$$\frac{d}{dt}\widetilde{\mathcal{E}}_k\big(u(t)\big) \leqslant C\langle t\rangle^{-1}\widetilde{\mathcal{E}}_k\big(u(t)\big)\big(\widetilde{E}_k^{\frac{1}{2}}\big(u(t)\big)+\widetilde{E}_k\big(u(t)\big)\big).$$

For the small solution, we also have

$$c\mathcal{E}_k\big(u(t)\big) \leqslant \widetilde{\mathcal{E}}_k\big(u(t)\big) \leqslant C\mathcal{E}_k\big(u(t)\big). \tag{4.10}$$

Combining all the estimates in this section, we finally obtain (4.1).

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